An Empirical Formula for Broad-Band SAR Calculations of Prolate Spheroidal Models of Humans and Animals

CARL H. DURNEY, MEMBER, IEEE, MAGDY F. ISKANDER, HABIB MASSOUDI, MEMBER, IEEE, AND C. C. JOHNSON

Abstract—An empirical relation for calculating approximate values of the average specific absorption rate (SAR) over a broad-frequency range for any prolate spheroidal model is derived for E-polarized incident plane waves. This formula provides a simple and inexpensive method for calculating the SAR for human and animal models, which otherwise requires complicated and expensive methods of calculation. The formula satifies the f^2 SAR behavior at lower frequencies, the resonance characteristic at intermediate frequencies, the $1/f$ behavior past resonance, and the dependence on the dielectric constant at the geometrical optics limits. An expression for the resonance frequency f_0 in terms of the dimensions of the model is also derived. The unknown expansion coefficients were determined by curve-fitting all the data available in the second edition of the _Radiofrequency Radiation Dosimetry Handbook_. Numerical results obtained from the empirical relations are generally in good agreement with those calculated by other methods. Limitations of the formula and suggestions for its improvement are also discussed.

I. Introduction

THEORETICAL STUDIES of the specific absorption rate (SAR) of electromagnetic (EM) energy by biological models have been of increasing interest in recent years because of the continuing need to evaluate the hazardous levels of the EM waves, and to refine the presently available safety standards. Of particular interest is the analysis of the prolate spheroidal models of man and animal that have been shown to give results that correlate well with those of more realistic models, as well as with the experimental results. Such calculations, however, are expensive and complex, particularly at frequencies near resonance. They also involve several theoretical methods, each one valid in a limited frequency range. Consequently, the *Radiofrequency Radiation Dosimetry Handbook* was published to provide average SAR values over a very wide frequency range for many animal and human models [1]. In many laboratory experiments, however, it often happens that an animal of different type or size than those specific cases included in the Handbook is used. In this case the researcher is left with the choice of either extrapolating the data available in the Handbook or calculating the SAR for the specific case of interest. The former is an inconvenient approximation of limited accuracy, while the latter is not only expensive and time consuming, but also beyond the interests or capabilities of many research organizations.

It is, therefore, desirable to have a simple method of calculating the average SAR over a broad range of frequencies. Such a method would be very valuable even if it were only to give results within 10 or 15 percent of the values calculated by more sophisticated techniques, since even the most accurate methods of calculation are based on models of humans and animals that are quite approximate. In the sequel, an empirical formula for calculating the average SAR over a very broad frequency range, for prolate spheroidal models of any human or animal, is developed using a combination of antenna theory, circuit theory, and curve fitting. The formula is derived only for the E-polarized incident plane waves, since it has been shown [1] that the highest SAR occurs for E polarization (incident electric field parallel to the major axis of the spheroid), and the case of greatest SAR is most important for evaluation of possible hazards.

II. Formulation of the Equation

Consider a prolate spheroidal model of a semimajor axis a and a semiminor axis b. In deriving a simple empirical formula that characterizes the average SAR as a function of frequency, it is important to take into account the following characteristics that are found to be common among the SAR's for free-space irradiation by an E-polarized incident plane wave.

1) For $a<\lambda/10$, where λ is the free-space wavelength, the SAR is approximately proportional to f^2. This f^2 behavior is exact for constant conductivity and can be derived from the long-wavelength approximation [2].
2) Each model has a resonant frequency at which the maximum absorption of the incident RF power occurs. The resonant frequency depends basically on the a and b of the model [3].
3) The SAR increases faster than f^2 just below resonance. Beyond resonance it is found from the experimental data that the average SAR decreases ap-

Manuscript received August 28, 1978; revised April 9, 1979. This work was supported by the USAF School of Aerospace Medicine, Brooks Air Force Base, under Contract F41609-76-C-0025/P00004.

C. H. Durney, M. F. Iskander, and H. Massoudi are with the Department of Electrical Engineering and the Department of Bioengineering, University of Utah, Salt Lake City, UT 84112.

C. C. Johnson, deceased, was with the Department of Electrical Engineering and the Department of Bioengineering, University of Utah, Salt Lake City, UT 84112.

0018-9480/79/0800-0758$00.75

proximately as $1/f$ [4]. The latter behavior is valid up to a high-frequency limit that varies with the dimensions of the model. For a spheroidal model of man size, for example, the $1/f$ behavior is expected to be valid for frequencies up to $6.7 f_0$, when f_0 is the resonant frequency [5].

4) At very high frequencies, where the wavelength is much smaller than the size of the irradiated object, the geometrical optics approximation is valid. In this case, the SAR does not depend on the frequency, but varies only with the permittivity ϵ.

A simple formula that satifies all of the above requirements is given by

$$\mathrm{SAR}=\frac{A_1 f^2/f_0^2\left[1+A_3(f/f_0)u(f-f_{01})+A_4A_5(f^2/f_0^2)u(f-f_{02})\right]}{(f^2/f_0^2)+A_2\left[f^2/f_0^2-1\right]^2}\ \mathrm{W/kg} \tag{1}$$

where the SAR is in watts per kilogram, $f_0<f_{01}<f_{02}$, A_1, A_2, A_3, and A_4 are functions of a and b, and A_5 is a function of ϵ. Also, $u(f-f_{0i})$ is a unit step function defined by

$$u(f-f_{0i})=\begin{cases}0, & f<f_{0i}\\ 1, & f>f_{0i}\end{cases}$$

where $i=1$ or 2. The step functions with f_{01} and f_{02} are used to provide the characteristic f^2 behavior at low frequencies, the $1/f$ behavior above resonance, and the frequency independent behavior at very high frequencies, as shown.

From (1) it is clear that if $f<f_{01}$, the SAR expression reduces to

$$\mathrm{SAR}=\frac{A_1 f^2/f_0^2}{(f^2/f_0^2)+A_2\left[f^2/f_0^2-1\right]^2}\ \mathrm{W/kg} \tag{2}$$

which is the same as that for a series RLC resonance circuit [6][1]. Also, if $f^2 \ll f_0^2$, it is easy to show that the SAR in (2) is proportional to f^2.

For $f_{01}<f<f_{02}$, the first unit step function will be nonzero, and hence (1) will reduce to

$$\mathrm{SAR}=\frac{A_1(f^2/f_0^2)\left[1+A_3 f/f_0\right]}{(f^2/f_0^2)+A_2\left[f^2/f_0^2-1\right]^2}. \tag{3}$$

For $A_3 f/f_0 \gg 1$ and $f^2/f_0^2 \gg 1$, (3) reduces to

$$\mathrm{SAR}=\frac{A_1A_3 f^3/f_0^3}{A_2 f^4/f_0^4}=\frac{A_1A_3 f_0}{A_2}\frac{1}{f} \tag{4}$$

which is the $1/f$ behavior described by Gandhi [5].

In the geometrical optics limit $f>f_{02}>f_{01}$, and hence all the terms in (1) will be included. For $f \gg f_0$, however, it can be easily shown that

$$\mathrm{SAR}=\frac{A_1A_4A_5}{A_2} \tag{5}$$

which is obviously frequency independent and depends only on ϵ as described by the function A_5.

The coefficients A_1, $A_2, \cdots, A_5$, f_{01}, and f_{02} were determined by least-square fitting all the data available in the *Radiofrequency Radiation Dosimetry Handbook* [1], as described below.

III. Numerical Procedures and Results

Since (1) is a nonlinear function of the parameters, the method of differential corrections together with Newton's iterative method was used [7]. The method of solution involves approximating (1) with a linear form that is convenient to solve iteratively. By estimating approximate values of the unknown coefficients $A_1^{(0)}$, $A_2^{(0)}, \cdots$, and of $f_{02}^{(0)}$ and expanding (1) in a Taylor's series with only the first-order terms retained, we obtain

$$\mathrm{SAR}\approx \mathrm{SAR}^{(0)}+\Delta A_1\left(\frac{\partial \mathrm{SAR}}{\partial A_1}\right)^{(0)}+\cdots+\Delta A_5\left(\frac{\partial \mathrm{SAR}}{\partial A_5}\right)^{(0)}+\cdots \tag{6}$$

where the superscript 0 is used to indicate values obtained after substituting the first guess $(A_1^{(0)}, A_2^{(0)}, \cdots, A_5^{(0)}, f_{01}^{(0)}, f_{02}^{(0)})$ for values of the unknown parameters into (1). Equation (6) is obviously a linear function of the correction terms ΔA_1, $\Delta A_2, \cdots, \Delta A_5$, and hence the least-square curve-fitting method can be used directly to determine these correction terms. The correction terms, when added to the first guess, give an improved approximation of the unknown coefficients; i.e., $A_1^{(1)}=A_1^{(0)}+\Delta A_1$, $A_2^{(1)}=A_2^{(0)}+\Delta A_2$, etc. When the improved estimates $A_1^{(1)}$, $A_2^{(1)}$, etc., are subsequently substituted as new estimates of the unknown coefficients, the Taylor's series reduces to

$$\mathrm{SAR}\approx \mathrm{SAR}^{(1)}+\Delta A_1\left(\frac{\partial \mathrm{SAR}}{\partial A_1}\right)^{(1)}+\cdots+\Delta A_5\left(\frac{\partial \mathrm{SAR}}{\partial A_5}\right)^{(1)}+\cdots$$

where $\mathrm{SAR}^{(1)}$ and its derivatives are obtained by substituting the values of $A_1^{(1)}$, $A_2^{(1)}, \cdots$, etc., into (1). Again, the correction terms ΔA_1, ΔA_2, etc., are determined using the least-square curve fitting method. The procedure is continued until the solution converges to within a specified accuracy. Numerical values of the coefficients obtained after curve-fitting 18 specific models available in the *Radiofrequency Radiation Dosimetry Handbook* are given in Table I.

The second step in the numerical procedure involves expressing the values of the expansion coefficients so

[1]The second term in the denominator of the equation on p. 413 of [6] should be squared. Also the resonant frequency for the given circuit parameters is negative.

TABLE I
NUMERICAL VALUES OF THE COEFFICIENTS EMPLOYED IN THE EMPIRICAL FORMULA (1)

Model	A_1	A_2	A_3	A_4	f_{o1}/f_o	f_{o2}/f_o
Average man	0.231	2.433	0.340	0.300	1.157	1.430
Skinny man	0.382	2.997	0.680	0.227	1.490	3.256
Fat man	0.1156	1.781	0.750	0.250	2.050	3.973
Average woman	0.239	2.325	0.559	0.242	1.370	2.620
Small woman	0.275	2.332	0.582	0.241	1.236	2.700
Large woman	0.188	2.190	1.000	0.350	1.740	5.500
10-year-old child	0.363	2.485	0.389	0.244	1.490	2.880
5-year-old child	0.385	2.330	0.376	0.228	1.040	2.690
1-year-old child	0.323	1.677	0.607	0.190	1.610	4.080
Rhesus monkey	0.282	1.000	0.592	0.160	2.340	4.687
Squirrel monkey	0.319	0.807	0.020	0.178	1.470	2.730
German shepherd	0.142	1.340	0.630	0.230	1.135	3.260
Brittany spaniel	0.174	1.250	0.600	0.205	1.300	3.510
Beagle	0.144	1.315	0.471	0.251	1.480	2.857
Guinea pig	0.513	0.707	0.212	0.117	1.800	3.333
Large rat	0.653	1.464	0.212	0.260	1.470	2.830
Medium rat	0.750	1.255	0.083	0.242	1.540	2.310
Small rat	1.050	1.000	0.040	0.180	0.950	1.630

obtained in terms of the a and b values of the 18 models used. Although each of the A's is expected to be a nonlinear function of a and b, the functions are chosen to be linear in the expansion coefficients. This allows a straightforward least-square curve fitting. The following are the expressions obtained by this procedure:

$$A_1 = -0.000994 - 0.01069a + 0.000172a/b + 0.000739(1/a) + 0.00566a/b^2 \tag{7}$$

$$A_2 = -0.00091 + 0.0414a + 0.39917a/b - 0.0012(1/a) - 0.00214a/b^2 \tag{8}$$

$$A_3 = 4.822a - 0.0835a/b - 8.733a^2 + 0.001575(a/b)^2 + 5.3688a^3 \tag{9}$$

$$A_4 = 0.3353a + 0.0753a/b - 0.804a^2 - 0.0075(a/b)^2 + 0.64a^3 \tag{10}$$

$$f_{01}/f_0 = -0.421a + 1.239a/b + 1.09a^2 - 0.2945(a/b)^2 + 0.0195(a/b)^3 \tag{11}$$

$$f_{02}/f_0 = 21.8a + 0.502a/b - 50.81a^2 - 0.068(a/b)^2 + 34.12a^3 \tag{12}$$

and

$$A_5 = |\epsilon/\epsilon_{20}|^{-1/4} \tag{13}$$

where a and b are in meters and ϵ_{20} is the dielectric constant of material having a permittivity equal to 2/3 that of muscle tissue at 20 GHz [1].

The resonance frequency f_0 can be estimated by assuming that the resonance is some combination of two conditions.

1) The length of the prolate spheroid is equal to $\lambda/2$ [1].

2) The circumference of the spheroid is equal to λ.

With these two assumptions, the following empirical formula for f_0 was obtained by curve-fitting data in the *Radiofrequency Radiation Dosimetry Handbook*:

$$f_0 = 2.75 \times 10^8 \left[8a^2 + \lambda^2(a^2 + b^2)\right]^{-1/2} \text{ Hz} \tag{14}$$

where a and b are in meters.

IV. RESULTS AND CONCLUSIONS

A comparison between values of f_0 obtained from (14) and those obtained using the extended boundary condition method are given in Table II. It is clear that (14) provides a quick and easy method of calculating the resonant frequency with approximately a 5-percent error at most.

Numerical results obtained from (1) are shown in Figs. 1 and 2, for the cases when the expansion coefficients are obtained from Table I, and from (7) to (13). As expected, the results are better with the coefficients in Table I, but the results from (7) and (13) give a very useful approximation. The poor agreement between the results given by the empirical relation and the Handbook values at the high-frequency end of the curve occurs because the frequency dependent values of ϵ were not used in A_5 in the empirical formula, but were included in the Handbook calculations. Much closer agreement would be obtained if the frequency dependence of ϵ were included in A_5.

One weakness in the empirical relation is that it often gives poor results at frequencies near f_{01}, which is caused by the abruptness of $u(f - f_{01})$. This limitation is not serious, however, since good results can be obtained by smoothing in the curve near f_{01}.

Gandhi [5] has given empirical relations for the resonant frequency and the SAR at resonance. A comparison with his results and some Handbook results are shown in Table III. Blank entries in the Handbook column indicate that calculations by the methods used for the Handbook are not yet possible for these particular cases, which illustrates another benefit of the empirical relation. That is, it can be used to obtain approximations for cases not included in the Handbook calculations. Although the error in these cases is difficult to estimate because more sophisticated calculations are not available, the results given by the two empirical methods show reasonable agreement. Examples of such results are shown in Figs. 3 and 4.

The empirical relation given here is useful for calculating the SAR for spheroidal sizes between a man and a rat, corresponding to the range of data used in the curve-fitting. However, it appears that the formula is also useful for some models smaller than a rat. For example, some calculations were made for models of mice and good results were obtained over a broad frequency band [1]. However, for some other spheroids smaller than rat-sized

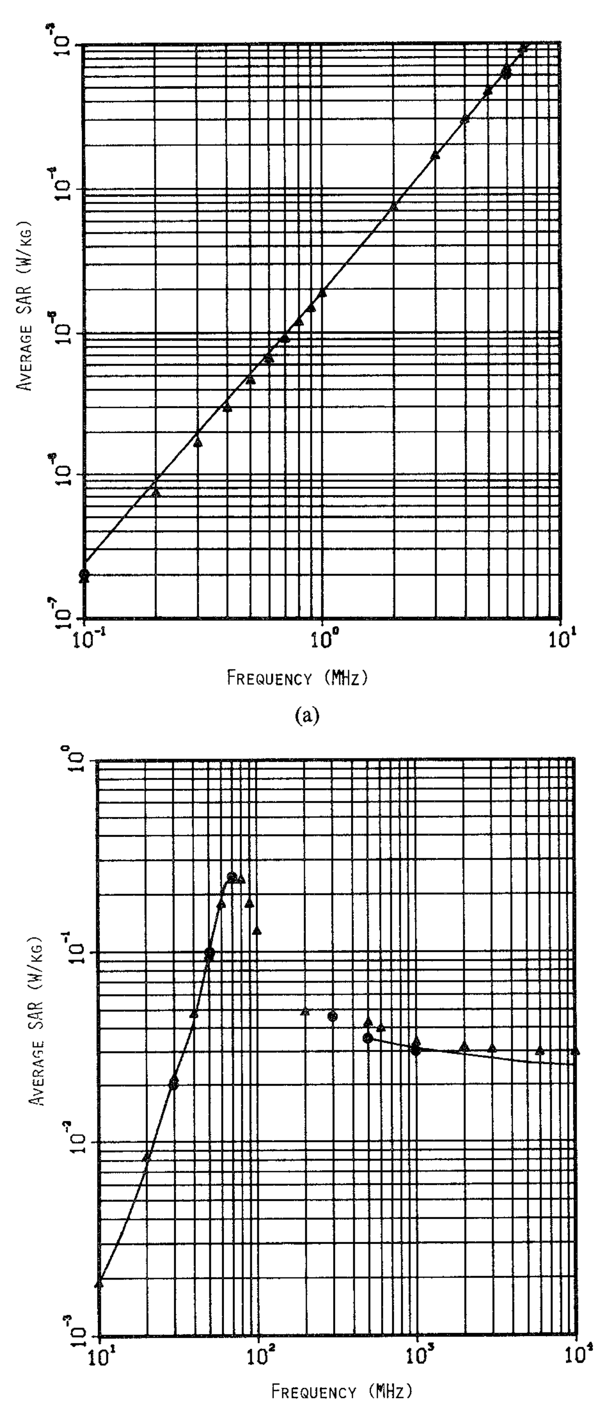

Fig. 1. Comparison of average SAR calculated by the empirical formula with the curve obtained by other calculations for a 70-kg average man. For the prolate spheroidal model, $a=0.875$ m and $b=0.138$ m. The incident plane wave is E-polarized with a density of 1 mW/cm^2. —— Calculated values [1]; ▲▲ empirical formula with the coefficients obtained from (7)–(14); and ●● empirical formula with the coefficients obtained from Table I.

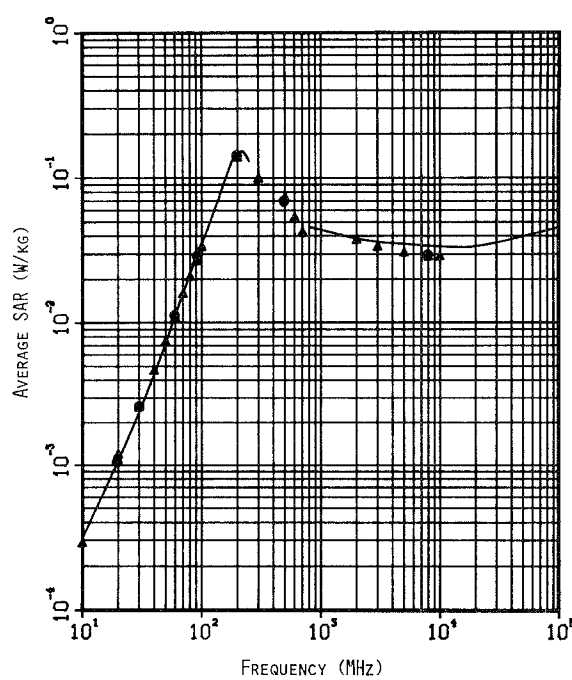

Fig. 2. Comparison of average SAR calculated by the empirical formula with the curve obtained by other calculations for a 13.5-kg beagle. For the prolate spheroid model, $a=28.5$ cm and $b=10.63$ cm. The incident plane wave is E-polarized with a power density of 1 mW/cm^2. —— Calculated values [1]; ▲▲ empirical formula with the coefficients obtained from (7)–(14); and ●● empirical formula with the coefficients obtained from Table I.

TABLE II
COMPARISON BETWEEN THE VALUES OF THE RESONANCE FREQUENCY f_0 OBTAINED USING THE EBCM [1] AND ESTIMATED FROM THE EMPIRICAL RELATION (14)

Model	a (m)	b (m)	f_o (MHz) From the RF Handbook [1]	f_o (MHz) From (14)	Percentage Error in Frequency
Average man	0.875	0.138	70	73.8	-5.6
Sitting rhesus monkey	0.200	0.0646	320	316.3	1.2
Squirrel monkey	0.115	0.0478	550	540.5	1.7
Beagle	0.285	0.1063	210	220.0	-4.8
Guinea pig	0.11	0.0355	600	575.6	4.1
Small rat	0.07	0.0194	950	910.2	4.2
Medium rat	0.10	0.0276	650	637.3	2.0
Large rat	0.120	0.0322	530	531.6	0.3

spheroids, the results were not as good. The accuracy of the empirical relation for the spheroids smaller than rat sized seems to depend strongly on the value of a/b.

For some spheroidal models, the SAR given by (1) at low frequencies does not fit as well as for other spheroidal models because of changes in ϵ with frequency. Since the coefficients in (1) were obtained by curve fitting without any attempt to include frequency dependence of the permittivity explicitly, fluctuations in SAR caused by a strong frequency dependence of the permittivity are not accounted for very well by (1). An example of this is shown in Fig. 5. Methods of including the frequency dependence of the permittivity explicitly in an empirical formula are being considered.

Although the empirical relation derived in this paper does have some limitations, as described above, it provides a very simple method for quickly calculating the approximate SAR as a function of frequency for prolate spheroidal models irradiated by plane waves with E-polarization, and as such should be very useful to those

TABLE III
COMPARISON BETWEEN THE SAR AND f_0 VALUES OBTAINED FROM [1], USING THE EMPIRICAL RELATION BY GANDHI [5] AND FROM (1)

Model	Empirical Relation by Gandhi [5]		From RF Handbook [1]		This Empirical Formula	
	SAR (W/kg)	F(MHz)	SAR (W/kg)	F(MHz)	SAR (W/kg)	F(MHz)
Average man	0.23	65.03	0.24	70	0.252	73.8
Average ectomorphic (skinny) man	0.34	64.68	—	—	0.382	73.6
Average endomorphic (fat) man	0.12	64.56	—	—	0.122	72.9
Average woman	0.22	70.97	—	—	0.242	80.2
Small woman	0.25	79.18	—	—	0.278	89.1
Large woman	0.18	65.78	—	—	0.193	74.5
10-year-old child	0.31	82.94	—	—	0.341	93.7
5-year-old child	0.34	101.56	—	—	0.378	115.3
1-year-old child	0.29	153.39	—	—	0.325	173.6
Rhesus monkey	0.38	285.	0.29	310	0.272	316.3
Squirrel monkey	0.40	496.	0.32	550	0.289	540.5
Baboon (Hamadryas)	0.22	166.4	—	—	0.153	184.3
German shepherd	0.21	126.73	—	—	0.146	141.3
Brittany spaniel	0.25	165.70	—	—	0.175	184.4
Beagle	0.20	200.01	0.15	200	0.142	220.0
Rabbit	1.33	284.72	—	—	0.953	322.6
Guinea pig	0.69	518.72	0.55	600	0.499	575.6
Small rat	1.48	815.96	1.1	950	1.06	910.2
Medium rat	1.04	569.44	0.78	750	0.749	637.3
Large rat	0.92	475.75	0.7	550	0.660	531.6
Small mouse	1.62	2109.0	—	—	1.18	2297.1
Medium mouse	2.03	1632.1	—	—	1.47	1803.9
Large mouse	1.92	1496.8	—	—	1.4	1663.0

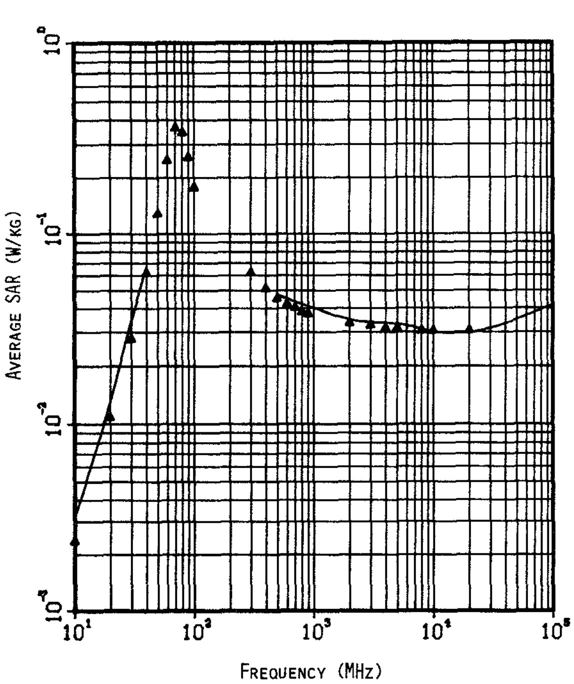

Fig. 3. Comparison of average SAR calculated by the empirical formula with the curve obtained by other calculations for a 47.18-kg skinny man. For the prolate spheroid model, $a=0.88$ m and $b=0.113$ m. The incident plane wave is E-polarized with a power density of 1 mW/cm^2. —— Calculated values [1], and ▲▲ are values obtained using the empirical formula.

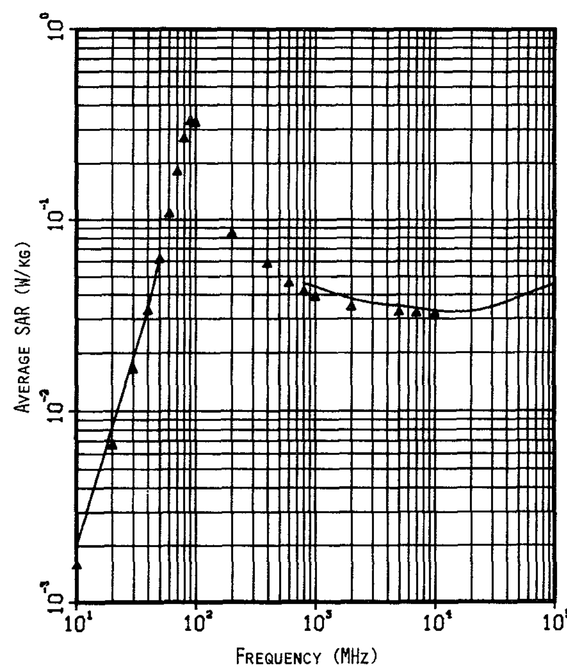

Fig. 4. Comparison of average SAR calculated by the empirical formula with the curve obtained by other calculations for a 32.2-kg 10-year-old child. For the prolate spheroid model, $a=0.69$ m and $b=0.106$ m. The incident plane wave is E-polarized with a power density of 1 mW/cm^2. —— Calculated values [1], and ▲▲ are values obtained using the empirical formula.

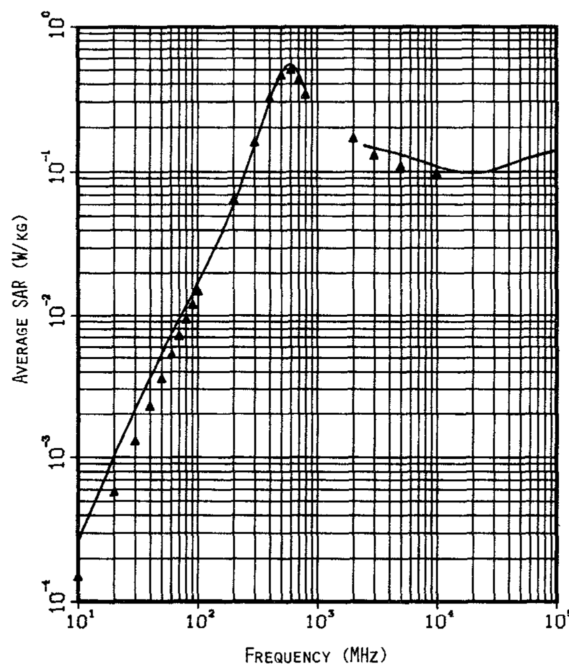

Fig. 5. Comparison of average SAR calculated by the empirical formula with the curve obtained by other calculations for a 0.58-kg guinea pig. For the prolate spheroid model, $a=11$ cm and $b=3.55$ cm. The incident plane wave is E-polarized with a power density of 1 mW/cm^2. —— Calculated values [1], and ▲▲ are values obtained using the empirical formula.

involved in microwave-biological research, as well as others interested in power absorption in lossy dielectric spheroids.

References

[1] C. H. Durney, C. C. Johnson, P. W. Barber, H. Massoudi, M. F. Iskander, J. L. Lords, D. K. Ryser, S. J. Allen, and J. C. Mitchell, *Radiofrequency Radiation Dosimetry Handbook*, Rep. SAM-TR-78-22, 1978, prepared by Elect. Eng. Dep., Univ. of Utah, Salt Lake City, and published by USAF School of Aerospace Medicine, Brooks Air Force Base, TX.

[2] C. H. Durney, C. C. Johnson, and H. Massoudi, "Long-wavelength analysis of planewave irradiation of a prolate spheroid model of man," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-23, pp. 246–254, 1975.

[3] P. W. Barber, "Resonance electromagnetic absorption by non-spherical dielectric objects," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-25, pp. 373–381, 1977.

[4] O. P. Gandhi, E. L. Hunt, and J. A. D'Andrea, "Deposition of electromagnetic energy in animals and in models of man with and without grounding and reflector effects," *Radio Sci.*, vol. 12, pp. 39–48, 1977.

[5] O. P. Gandhi and M. J. Hagmann, "Some recent results on deposition of electromagnetic energy in animals and models of man," presented at the Int. Symp. on biological effects of electromagnetic waves, Airlie, VA, Oct. 30–Nov. 4, 1977. See also, *Radio Sci.*, vol. 12, no. 6(S), pp. 39–48, 1977.

[6] H. R. Kucia, "Accuracy limitation in measurements of HF field intensities for protection against radiation hazards," *IEEE Trans. Instrum. Meas.*, vol. IM-21, pp. 412–415, 1972.

[7] L. H. Lafara, *Computer Methods for Science and Engineering*. New York: Hayden, 1973, ch. 3.